

ORTHOCENTRIC PROPERTIES OF THE PLANE n -LINE*

BY

F. MORLEY

In continuation of a memoir in these Transactions (vol. 1, p. 97) I consider the problem :

To find for n lines of a plane natural metrical analogues of the elementary facts that the perpendiculars of 3 lines meet at a point (the orthocenter of the 3-line) and that the orthocenters of the 3-lines contained in a 4-line lie on a line.

I apply first to the special case of a 4-line the treatment sketched in § 7 of the memoir cited ; this affords suggestions for the general case.

§ 1. *The Deltoid.*

To discuss with the minimum of trouble the metrical theory of a 4-line we should take, according to our purpose, the lines as tangents either of a parabola or of a hypocycloid of class three. We want here the latter curve. As it is to most geometers an incidental stationary thing and not a weapon, I will treat it *ab initio*. And as it is at least as good as other curves which have a given name I will call it a deltoid. It is hardly necessary to remark that it is the metrically normal form of the general rational plane curve of class three with isolated double line.

Denote by t a turn or a complex number of absolute value 1 ; and think of t as a point on the unit circle. We consider three points t_1, t_2, t_3 on the circle subject to the condition

$$(1) \quad s_3 \equiv t_1 t_2 t_3 = 1.$$

With this triad we associate a point x by the equation

$$(2) \quad x = t_1 + t_2 + t_3 \equiv s_1,$$

which carries with it the conjugate equation

$$y = 1/t_1 + 1/t_2 + 1/t_3 = t_2 t_3 + t_3 t_1 + t_1 t_2 \equiv s_2.$$

We have from (1) and (2)

$$(3) \quad x = t_1 + t_2 + 1/t_1 t_2.$$

* Presented to the Society at the Evanston meeting, September 2-3, 1902. Received for publication September 3, 1902.

Herein let t_1 be fixed, t_2 variable. Since $x = t + 1/t$ is that simplest of all hypocycloids, the segment of a line, so also is (3) the segment of a line, of constant length 4, with center t_1 and ends where

$$D_{t_2} x = 0,$$

or

$$t_1 t_2^2 = 1,$$

or

$$x = t_1 \pm 2/\sqrt{t_1}.$$

When t_1 varies, the deltoid is described by the motion of this variable segment; the ends move on the curve, and the segment touches the curve. And any point inside the curve is given by (3) as the intersection of 2 segments; the points of the curve itself are given when the segments are brought to coincidence, that is, are

$$(4) \quad x = 2t + 1/t^2.$$

This equation expresses that while a point $2t$ is describing a circle of radius 2 another point x moves round it with an angular velocity opposite in sense and twice as great; thus the cycloidal nature of the curve is apparent.

A point of the curve and also a line of the curve is named by its parameter t ; thus the point t of the curve is given by (4), and the line t from (1) and (2) by

$$(5) \quad xt^2 - yt = t^3 - 1.$$

x and y being always conjugate coördinates.

Now two lines t_1 and t_2 meet at

$$x_1 = t_1 + t_2 + 1/t_1 t_2.$$

Symmetrize this equation for 3 lines of the curve by writing it

$$(6) \quad x = s_1 - t_3 + t_3/s_3.$$

Omitting the suffix of t_3 we have the map-equation of the circumcircle. Hence: The circumcenter of the 3-line is s_1 , and the circumradius is $|1 - 1/s_3|$ or $|1 - s_3|$.

The mean point of

$$t_1 + t_2 + 1/t_1 t_2 \quad \text{and} \quad t_1 + t_3 + 1/t_1 t_3$$

is

$$m = \frac{1}{2} [s_1 + t_1 + (s_1 - t_1)/s_3].$$

Hence the center of the circle which bisects the sides, the Feuerbach or nine-point circle, is

$$(7) \quad c = \frac{1}{2} (s_1 + s_1/s_3).$$

The perpendicular from $t_1 + t_2 + 1/t_1 t_2$ on the line t_3 is

$$xt_3 + y = t_3(t_1 + t_2 + 1/t_1 t_2) + 1/t_1 + 1/t_2 + t_1 t_2 = s_2 + t_3 s_1/s_3.$$

Hence the orthocenter is

$$p = s_1/s_3,$$

as could of course be inferred from the fact that

$$\text{orthocenter} + \text{circumcenter} = 2 \times \text{Feuerbach center}.$$

Since then $|p| = |s_1|$ we have:

THEOREM 1. *The center of a deltoid, of which three lines are given, is equidistant from the circumcenter and orthocenter of the 3-line.*

The centre of the deltoid which touches 4 lines is thus determined as the intersection of 4 lines, one for each of 3 of the given 4.

The Feuerbach center for 3 of 4 lines is given by

$$2c = (s_1 - t_4)(1 + t_4/s_4), \quad (s \text{ for } 4),$$

s_1 and s_3 being replaced in (7) by symmetric functions of t_1, t_2, t_3, t_4 . Thus the 4 such centers are included in

$$2x = (s_1 - t)(1 + t/s_4),$$

whose conjugate is

$$2y = (s_3/s_4 - 1/t)(1 + s_4/t) = (s_3/t - s_4/t^2)(1 + t/s_4).$$

But

$$t^2 - s_1 t + s_2 - s_3/t + s_4/t^2 = 0.$$

Therefore

$$2(tx + y) = s_2(1 + t/s_4),$$

a line through $\frac{1}{2}s_2/s_4$, perpendicular to the line t of the curve. Hence

THEOREM 2. *If from the Feuerbach center of any 3 of 4 lines a perpendicular be drawn to the remaining line, the 4 perpendiculars meet at a point, namely $\frac{1}{2}s_2/s_4$.*

§ 2. Extension to the n -line.

I employ now the notation of § 2 of the memoir cited, namely I write a line l_a in the form

$$xt_a + y = x_a t_a = y_a.$$

Denote the characteristic constants by α_a ,

$$\alpha_a = \sum \frac{x_1 t_1^{n-a}}{(t_1 - t_2) \cdots (t_1 - t_n)}, \quad (a = 1, 2, \dots, n)$$

and their conjugates by b_a , so that

$$b_a = (-)^{n-1} s_n a_{n+1-a}$$

where $s_n = t_1 t_2 \dots t_n$, and in general $s_a = \sum t_1 t_2 \dots t_a$ for n t 's.

The circumcenter of a 3-line is a_1 . The mean point of the joins of l_1, l_2 and l_1, l_3 is given by

$$\begin{aligned} 2m &= \frac{x_1 t_1}{t_1 - t_2} + \frac{x_2 t_2}{t_2 - t_1} + \frac{x_1 t_1}{t_1 - t_3} + \frac{x_3 t_3}{t_3 - t_1} \\ &= \frac{x_1 t_1 (2t_1 - s_1 + t_1)}{(t_1 - t_2)(t_1 - t_3)} + \frac{x_2 t_2 (2t_2 - s_1 + t_1)}{(t_2 - t_1)(t_2 - t_3)} + \frac{x_3 t_3 (2t_3 - s_1 + t_1)}{(t_3 - t_1)(t_3 - t_2)}, \end{aligned}$$

and therefore the nine-point circle is

$$(8) \quad 2m = 2a_1 - s_1 a_2 + a_2 t,$$

and its center is

$$2c = 2a_1 - s_1 a_2.$$

Hence the orthocenter is

$$(9) \quad p = 2c - a_1 = a_1 - s_1 a_2.$$

The line about the points a_1 and p (which is the locus of centers of inscribed deltoids) is, when τ is any turn,

$$(10) \quad x - a_1 + s_1 a_2 = (x - a_1) \tau.$$

This for 3 of 4 lines is

$$x - a_1 + t a_2 + (s_1 - t)(a_2 - t a_3) = (x - a_1 + t a_2) \tau,$$

$$\text{or} \quad x - a_1 + s_1 a_2 - t(s_1 - t)a_3 = [x - a_1 + s_1 a_2 - (s_1 - t)a_2] \tau.$$

Now since for 4 lines a_3/a_2 is a turn we can equate $t a_3$ to $a_2 \tau$; that is, whatever turn t may be, the line passes through the point

$$(11) \quad p_1 = a_1 - s_1 a_2.$$

This is then the center of the inscribed deltoid of the 4-line.

For 4 of 5 lines this point is given by

$$p = a_1 - t a_2 - (s_1 - t)(a_2 - t a_3) = a_1 - s_1 a_2 + t(s_1 - t)a_3,$$

or the conjugate equation

$$q = b_1 - \frac{s_4}{s_5} b_2 - \frac{1}{t} \left(\frac{s_4}{s_5} - \frac{1}{t} \right) b_3 = s_5 a_5 - s_4 a_4 + \left(\frac{s_4}{s_5} - \frac{s_5}{t^2} \right) a_3.$$

But for 5 things

$$t^3 - t^2 s_1 + t s_2 - t s_3 + s_4/t - s_5/t^2 = 0.$$

Therefore

$$(12) \quad t(p - a_1 + s_1 a_2) - (q - s_5 a_5 + s_4 a_4) = (ts_2 - s_3) a_3,$$

that is, the line joining the p of 4 lines to the point $a_1 - s_1 a_2 + s_2 a_3$ is perpendicular to the remaining line; or

THEOREM 3. *If from the center of the inscribed deltoid of 4 of 5 lines a perpendicular be drawn to the line left out, the 5 perpendiculars meet at a point; namely the point*

$$p_2 \equiv a_1 - s_1 a_2 + s_2 a_3.$$

Call this point the *first orthocenter* of the 5-line.

For 5 of 6 lines this point is

$$\begin{aligned} x &= a_1 - ta_2 - (s_1 - t)(a_2 - ta_3) + (s_2 - ts_1 + t^2)(a_3 - ta_4) \\ &= a_1 - s_1 a_2 + s_2 a_3 - t(s_2 - ts_1 + t^2) a_4 \\ &= p_2 - t(s_2 - ts_1 + t^2) a_4, \end{aligned}$$

or, if the conjugates be written,

$$y = q_2 + (s_4/t - s_5/t^2 + s_6/t^3) a_3.$$

But for 6 things

$$t^3 - t^2 s_1 + ts_2 - s_3 + s_4/t - s_5/t^2 + s_6/t^3 = 0.$$

Hence

$$(13) \quad \frac{x - p_2}{a_4} - \frac{y - q_2}{a_3} = -s_3,$$

that is to say

THEOREM 4. *The first orthocenters of the 5-lines included in a 6-line lie on a line.*

The argument is clearly general, so that if the point $p_2 \equiv a_1 - s_1 a_2 + s_2 a_3$ be constructed for a 6-line, the perpendiculars from such point for 6 of 7 lines on the line left out meet at a point, and for 7 of 8 lines these points lie on a line; and so on. Briefly, we have found an *orthocenter* for an odd number of lines, a *directrix* for an even number.

§ 3. Construction of a series of points.

The points to which attention is thus directed belong, for a given n -line, to the series

$$(14) \quad p_0 = a_1, \quad p_1 \equiv a_1 - s_1 a_2, \quad p_2 \equiv a_1 - s_1 a_2 + s_2 a_3, \text{ etc.}$$

Their construction is merely a matter of centroids, or centers of gravity. For we regard as known in an n -line:

and extend this. We have for an n -line,

$$\begin{aligned} x - p_2 + t(s_2 - ts_1 + t^2)a_4 &= \{x - p_1 - t(s_1 - t)a_3\}z \\ &= \{x - p_2 + (s_2 - ts_1 + t^2)a_3\}z, \end{aligned}$$

whence, as before, the point p_2 of the n -line has its distances from the points p_2 and p_1 of any included $n - 1$ lines in the fixed ratio $|a_4/a_3|$. And so in general:

THEOREM 5. *The point p_a of an n -line has its distance from the points p_a and p_{a-1} of an included $(n - 1)$ -line in the fixed ratio $|a_{a+2}/a_{a+1}|$.*

Since $b_a = (-)^{n-1} s_{n+1-a}$, this fixed ratio is unity when

$$\alpha + 2 + \alpha + 1 = n + 1,$$

or

$$\alpha + 1 = \frac{1}{2}n,$$

that is, in a $2n$ -line the point p_{n-1} is equidistant from the points p_{n-1} and p_{n-2} of any included $(2n - 1)$ -line.

Regarding the lengths $|a_a|$ as known, we have in Theorem 5 a construction for the points p_a for n -lines, when the points p_a for $(n - 1)$ -lines are known.

§ 4. The curve Δ^{2n-1} .

The peculiar appropriateness of the deltoid for the metrical theory of four lines makes it desirable to have an analogous curve for $2n$ -lines. Such a curve is

$$(16) \quad \begin{aligned} (-)^n (xt^n - yt^{n-1}) &= t^{2n-1} - 1 - (s_1 t^{2n-2} - s_{2n-2} t) \\ &\quad + \cdots + (-)^n (s_{n-2} t^{n+1} - s_{n+1} t^{n-2}), \end{aligned}$$

where c_a and c_{2n-1-a} are conjugate. This is a curve Δ^{2n-1} of class $2n - 1$, order $2n$, with a line equation of the type

$$(17) \quad \xi^n \eta^n = \text{form in } \xi, \eta \text{ of order } 2n - 1.$$

For clearness I will take the case $n = 3$, next to the case $n = 2$ of § 1; the generalizations are immediate.

Any 6 lines are lines of a curve Δ^5 ,

$$(18) \quad -(x^3 - yt^2) = t^5 - 1 - (s_1 t^4 - s_4 t).$$

The map-equation of the curve is

$$(19) \quad -x = 3t^2 + 2t^{-3} - (2s_1 t - s_4 t^{-2}).$$

Thus the curve is derived by addition from 2 concentric cycloids.

So Δ^{2n-1} is derived by addition from $n - 1$ concentric cycloids; those points being added at which the tangents are parallel.

Let the common center of the cycloids be called the center of Δ^{2n-1} .

The cusps of Δ^5 are given from (19) by

$$6(t - t^4) - 2(s_1 - s_4 t^{-3}) = 0,$$

or

$$3(t^5 - 1) = s_1 t^4 - s_4 t.$$

Hence the cusp-tangents are such that

$$(20) \quad 3(xt^3 - yt^2) = 2(s_1 t^4 - s_4 t),$$

that is :

There are 5 cusp-tangents of Δ^5 ; they touch a concentric Δ^3 . And so

THEOREM 6. *There are $2n - 1$ cusp-tangents of Δ^{2n-1} ; they touch a concentric Δ^{2n-3} .*

Consider the common lines of Δ^5 and any Δ^3 ,

$$xt^2 - yt = \alpha t^3 + \beta t^2 - \gamma t - \delta.$$

There are 5 common lines, and they are given by

$$(21) \quad t^5 - 1 - (c_1 t^4 - c_4 t) + t(\alpha t^3 + \beta t^2 - \gamma t - \delta) = 0.$$

Hence the center of the Δ^3 is

$$\beta = \sum t_1 t_2 = s_2$$

where

$$s_5 = 1,$$

that is, the center of the deltoid touching 4 lines of Δ^5 is

$$(22) \quad x = s_2 + s_1/s_4 \quad (s \text{ for } 4).$$

The perpendicular on a fifth line of Δ^5 is

$$xt + y = t(s_2 + s_1/s_4) + s_2/s_4 + s_3 = s_3 + s_2 t/s_5 \quad (s \text{ for } 5).$$

Hence the first orthocenter of 5 lines is s_2/s_5 .

For 5 of 6 lines this point is

$$x = (t^2 - ts_1 + s_2)t/s_6$$

or

$$y = s_6/t^3 - s_5/t^2 + s_4/t,$$

whence the 6 first orthocenters lie on the line

$$(23) \quad s_6 x + y = s_3.$$

The line of Δ^3 perpendicular to a fifth line of Δ^5 is

$$xt^2 + yt + \alpha t^3 - \beta t^2 - \gamma t + \delta = 0,$$

where

$$c_1 - \alpha = s_1, \quad \beta = s_2, \quad \gamma = s_3, \quad c_4 - \delta = s_4, \quad 1 = s_5;$$

or is

$$xt^2 + yt + (c_1 - s_1 - 1/s_4)t^3 - (s_2 + s_1/s_4)t^2 - (s_3 + s_2/s_4)t + c_4 - s_4 - s_3/s_4 = 0 \quad (s \text{ for } 4),$$

or

$$(24) \quad xt^2 + yt + c_1 t^3 - s_2 t^2 - s_4 + c_4 - s_3 t/s_5 - s_1 t^3/s_5 = 0 \quad (s \text{ for } 5).$$

That is :

THEOREM 7. *If of the deltoid touching any 4 of 5 lines we draw the line perpendicular to the omitted line, the 5 perpendiculars touch a deltoid.*

The center of this deltoid is s_2 . The first orthocenter was s_2/s_5 . These are strokes of equal size. Hence :

THEOREM 8. *The locus of centers of curves Δ^5 of which 5 lines are given is a line. And so for Δ^{2n-1} .*

The curve Δ^{2n-1} does then for $2n$ lines precisely what the deltoid Δ^3 does for 4 lines ; it replaces Clifford's n -fold parabola for metrical purposes. We have proved by its means the theorems of § 2 over again with additions ; in particular we have assigned a meaning to the point p_1 of a 5-line or p_{n-2} of a $(2n-1)$ -line, for this is readily identified as the point s_2 of 5 lines of Δ^5 or s_{n-1} of $2n-1$ lines of Δ^{2n-1} . But at present I regard the use of this curve as more limited than the method of the a 's, to which I now return.

§ 5. *The second circle of an n -line.*

A curve of order n , whose highest terms in conjugate coördinates are

$$tx^n + y^n,$$

has its asymptotes apolar to the absolute points IJ , that is, these asymptotes form an equiangular polygon. Such a curve depends on $\frac{1}{2}n(n+1) + 1$ constants, and therefore a pencil can be drawn through $\frac{1}{2}n(n+1)$ points in general, and the pencil determines $n^2 - \frac{1}{2}n(n+1)$, or $\frac{1}{2}n(n-1)$ other points. In the pencil are the imaginary curves

$$x^n + an^{n-1} + \dots + a'y^{n-1} + \dots = 0$$

and

$$y^n + by^{n-1} + \dots + b'y^{n-1} \dots = 0,$$

the pencil itself is

$$(25) \quad tn^n + y^n + (ta + b')x^{n-1} + \dots = 0,$$

the polar line of I is

$$ntx + ta + b' = 0,$$

and therefore the polar lines of I and J meet on a circle.*

Let now the $\frac{1}{2}n(n+1)$ points be the joins of $n+1$ lines. We shall call the circle the *second circle* of the $(n+1)$ -line; the first circle being the center-circle (l. c., p. 99).

For a 3-line the second circle is the Feuerbach circle.

Now calculate the center and radius of the second circle of the n -line

$$xt_a + y = x_a t_a = y_a \quad (\alpha = 1, \dots, n).$$

The pencil is

$$\sum_a \frac{A_a}{nt_a + y - y_a} = 0.$$

The highest powers arise from

$$\sum_a \frac{A_a}{xt_a + y},$$

and are to be $tx^{n-1} + y^{n-1}$, so that, if $y/x = \lambda$,

$$(26) \quad t + \lambda^{n-1} = \sum A_1(t_2 + \lambda)(t_3 + \lambda) \cdots (t_n + \lambda);$$

whence

$$t = \sum A_1 t_2 \cdots t_n,$$

$$t + (-t_1)^{n-1} = A_1(t_2 - t_1)(t_3 - t_1) \cdots (t_n - t_1).$$

Operating with D_x^{n-2} on

$$\sum A_1(xt_2 + y - y_2) \cdots (xt_n + y - y_n),$$

we have for the polar line of I

$$(n-1)tx = \sum A_1 t_2 t_3 \cdots t_n (x_2 + x_3 + \cdots + x_n),$$

or if

$$\sum x_a = ng_{n-1},$$

$$(n-1)tx = ntg_{n-1} - \sum n_1 \frac{\{t + (-t_1)^{n-1}\} t_2 \cdots t_n}{(t_2 - t_1)(t_3 - t_1) \cdots (t_n - t_1)};$$

or since

$$\sum x_1 \frac{t_2 \cdots t_n}{(t_2 - t_1) \cdots (t_n - t_1)} = a_1 - s_1 a_2 + \cdots + (-)^{n-1} s_{n-1} a_n = p_{n-1},$$

the second circle is

$$(27) \quad (n-1)x = ng_{n-1} - p_{n-1} - s_n a_2 / t.$$

* Cf. J. H. GRACE, Proceedings London Mathematical Society, vol. 33 (1900), p. 194.

Hence its radius is

$$\left| \frac{s_n a_2}{n-1} \right|, \quad \text{i. e.,} \quad \frac{1}{n-1} |a_2|, \text{ or:}$$

THEOREM 9. *The radius of the second circle of an n -line is $1/(n-1)$ of the radius of the first circle.*

Also its center is given by

$$(n-1)c = ng_{n-1} - p_{n-1},$$

or explicitly by

$$(28) \quad (n-1)c = (n-1)a_1 - (n-2)s_1a_2 + \cdots - (-)^n s_{n-2}a_{n-1}.$$

Omitting now the n th line we have for the second center of the rest

$$(n-2)c' = (n-1)g'_{n-2} - p'_{n-2},$$

$$\text{whence} \quad (n-1)c - (n-2)c' = x_n - (p_{n-1} - p'_{n-2}).$$

Here

$$\begin{aligned} p'_{n-2} &= a_1 - ta_2 - (s_1 - t)(a_2 - ta_3) + \cdots \\ &\quad + \{t^{n-2} - s_1 t^{n-3} + \cdots + (-)^n s_{n-2}\}(a_{n-1} - ta_n) \\ &= p_{n-2} - a_n \{t^{n-1} - s_1 t^{n-2} + \cdots + (-)^n t s_{n-2}\} \\ &= p_{n-1} - a_n \{t^{n-1} - s_1 t^{n-2} + \cdots - (-)^n s_{n-1}\} \\ &= p_{n-1} + (-)^n a_n s_n / t \\ &= p_{n-1} - b_1 / t. \end{aligned}$$

Therefore

$$(n-2)c' = (n-1)c - x_n + b_1/t_n.$$

But the reflexion r' of the first center a_1 in the omitted line is $x_n - b_1/t_n$. Hence

$$(29) \quad (n-2)c' = (n-1)c - r',$$

whence it follows at once that:

THEOREM 10. *If from the second center of each $(n-1)$ -line of an n -line a perpendicular be drawn to the omitted line, these perpendiculars meet at a point; the point is the external center of similitude of the first and second circles of the n -line.*

The point h so found is given by

$$(30) \quad \begin{aligned} (n-2)h &= (n-1)c - a_1 = (n-2)(a_1 - s_1 a_2) + (n-3)s_2 a_3 \\ &\quad - (n-4)s_3 a_4 + \cdots - (-)^n s_{n-2} a_{n-1}. \end{aligned}$$

Whereas the orthocenter of § 2 applied only to an odd number of lines the present one applies to any number. We have then for an odd n -line *two* solutions of our problem, except when $n = 3$, in which case the points h and p coincide.

But when we have two orthocenters, we have a whole line of orthocenters, since evidently a perpendicular to one of the n lines, dividing the join of the two known points of the remaining $n - 1$ -lines in a fixed ratio, will divide the join of the two orthocenters in the same fixed ratio.

Thus we have found, *for an even number of lines, one orthocenter; for an odd number of lines, a line of orthocenters; for an even number of lines, one directrix.*

KNOWLTON, P. Q.,

July, 1902.
